

Collisional relaxation of two-dimensional gravitational systems

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Abstract

Systems with long range interactions present generically the formation of quasi-stationary long-lived non-equilibrium states. These non-Boltzmann states relax to Boltzmann equilibrium following a dynamic which is not well understood. In this letter we derive a simple approximate kinetic equation for the relaxation process in a two-dimensional inhomogeneous self-gravitating particle system, obtaining a Fokker – Planck equation for the velocity distribution with explicit analytical diffusion coefficients. Performing molecular dynamics simulations using the full dynamics and comparing them with the evolution predicted by the Fokker – Planck equation, we observe a good agreement with the model for all the duration of the relaxation, from the formation of the Quasi-Stationary state to thermal equilibrium. During all this process we observe a scaling of the relaxation time proportional to the number of particles in the system.

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Systems of particles with long range interactions are those which inter-particle potential at large separation decays slower than the dimension d of space, i.e., $v(r \rightarrow \infty) \sim 1/r^\gamma$ with $\gamma \leq d$. There are many examples in nature such as self-gravitating systems in the cosmological and astrophysical context (the large structure of the universe, galaxies, etc), interaction between vortices in two-dimensional hydrodynamics, cold classical atoms or capillary interactions between colloids or granular media (for a review see e.g. [1]). These kinds of systems present very particular properties in thermal equilibrium, such that negative micro-canonical specific heat or inequivalence of statistical ensembles. Their dynamics is also peculiar compared to short range systems: in a first stage there is the generic formation in a few characteristic times τ_{dyn} of a long-lived non-equilibrium state — during the so-called *violent relaxation* process. A typical example of such quasi-stationary states (hereafter QSS) are galaxies or young globular clusters. Then, a comparatively very slow relaxation to thermodynamical equilibrium occurs — called *collisional relaxation* — in a timescale of order $\tau_{coll} \sim N^\delta \tau_{dyn}$, where N is the number of particles and $\delta \geq 1$ depends on the system studied.

The mechanism of collisional relaxation is still not well understood. In the context of gravitational systems, Chandrasekhar found theoretically, in a seminal work [2], an estimate of the relaxation time for gravitational systems in three dimensions. He considered an homogeneous system and computed the change in velocity due to successive independent collisions of a test particle in a stationary macroscopic configuration. Because of the hypothesis of homogeneity there is no macroscopic scale in the system, which led to an ongoing controversy about the value of the maximal impact parameter of the collisions and in particular how it should scale with N [3–6]. Following this, several studies considered collective ef-

fects (e.g. [7]), but still in homogeneous configurations. An explicit theoretical description of the collisional relaxation in inhomogeneous systems is still lacking, despite recent progress in this direction [8, 9].

The collisional relaxation has also been studied numerically, for a wide variety of systems. In $d = 1$ dimensions for one-dimensional gravity, a scaling of $\tau_{coll} \sim N \tau_{dyn}$ has been measured for the full relaxation process [10], and in the Hamiltonian Mean Field model the scaling has been found to be dependent on the initial condition: $\tau_{coll} \sim N \tau_{dyn}$ [11], $\tau_{coll} \sim N^{1.7} \tau_{dyn}$ [11] or $\tau_{coll} \sim \exp(N) \tau_{dyn}$ [12]. For dimensions larger than $d = 1$, the relaxation has been estimated studying — for numerical reasons — only its early stage, i.e., for times in which the QSS is weakly perturbed (see e.g. [13, 14]), or performing simulations with a simplified dynamics. For gravity in two-dimensions, in simulations performed imposing radial symmetry, it has been observed $\tau_{coll} \sim N^{1.35} \tau_{dyn}$ [15]. In $d = 3$ dimensions, the Chandrasekhar scaling $\tau_{coll} \sim N / \ln N \tau_{dyn}$ has been verified for gravity (e.g. [4, 13, 16]) and for power-law potential $u(r) = 1/r^\gamma$, for which has been found $\tau_{coll} \sim N \tau_{dyn}$ if $\gamma < 1$, see [14, 16].

In this letter, we present an analytical and numerical study of the collisional relaxation of a self-gravitating system in $d = 2$ dimensions. The interacting potential — solution of the Poisson equation in $d = 2$ dimensions — is $u(r) = g \ln(r)$, where g is the coupling constant. It is an attractive model to understand the realistic $d = 3$ gravity: it presents the same mechanism of collisions as in $d = 3$ (which is not the case for models in $d = 1$), the system is self-confined (it is not necessary to confine it artificially in a box), thermal equilibrium properties are easily calculated and numerical simulations are easier to perform than in $d = 3$. In this work, we simulated the relaxation dynamics of the system, for the whole time range between the QSS and the final thermal equilibrium. The

simulations are very well described by an approximate kinetic equation we derive in what follows.

Theory.— We model the generic evolution of the system using the Boltzmann equation for the one point probability density function $f(\mathbf{r}, \mathbf{v}, t)$. We can write it formally as

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{F}[f] \cdot \frac{\partial f}{\partial \mathbf{v}} = \Gamma_c[f], \quad (1)$$

where $\Gamma_c[f]$ is the collision operator. During the relaxation process, the system reaches first a QSS and then evolves (comparatively slowly) through an infinity sequence of QSS, in which $\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{F}[f] \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$. To make Eq. (1) tractable analytically, we neglect its second and third term, which implies not taking collective effects into account. We focus on the evolution of the velocity distribution $s(\mathbf{v}, t) = \int d^2r f(\mathbf{r}, \mathbf{v}, t)$. We integrate Eq. (1) over the positions, obtaining, in the approximation described above

$$\frac{\partial s}{\partial t} = \int d^2r \Gamma_c[f]. \quad (2)$$

In the same manner as in the most studied $d = 3$ case, the relaxation is dominated by *soft collisions* (see e.g. [17]), which implies that the change in velocity of a particle in a collision is much smaller than its velocity. Moreover, it has been shown that, for times larger than one orbital period, the force correlation function decays rapidly (e.g. as $\sim 1/t^5$ for gravity in $d = 3$ [18]). We may then consider that collisions are independent and the use of a Fokker-Planck approximation of Eq. (2) is therefore justified (see e.g. [19]), which can be written as

$$\frac{\partial s(\mathbf{v}, t)}{\partial t} = \frac{\partial}{\partial v_i} [D_{v_i} s(\mathbf{v}, t)] + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} [D_{v_i v_j} s(\mathbf{v}, t)]. \quad (3)$$

The terms $D_{v_i}(\mathbf{v}, t)$ and $D_{v_i v_j}(\mathbf{v}, t)$ are the diffusion coefficients, which are defined as the average change of the velocity of the particles per unit of time, $\langle \Delta v_i \rangle / \Delta t$ and $\langle \Delta v_i \Delta v_j \rangle / \Delta t$, respectively.

We estimate first the change in velocity of a particle due to a “collision” with a single particle. In the context of long range systems, we define a “collision” between two particles as the process in which they cross each other in one orbital period. The spatial density distribution at thermal equilibrium of a self-gravitating system in $d = 2$ generates a (mean-field) gravitational potential of the form $\Psi(r) = 2 \ln(\lambda^2 + Ar^2) / \beta$, where β is the inverse temperature and λ and A are two constants which depend on the number of particles N and on the total energy of the system (see e.g. [15]). For $r \lesssim \lambda$ (which corresponds to a scale in which are included half of the particles), the potential is harmonic, i.e., $\Psi(r) \simeq 4 \ln \lambda + \omega^2 r^2 / 2$ (where $\omega = 4A / \beta \lambda^2$). It has been shown numerically that this is also true in $d = 3$ for a wide set of initial conditions [20]. Assuming that the potential has this approximate form

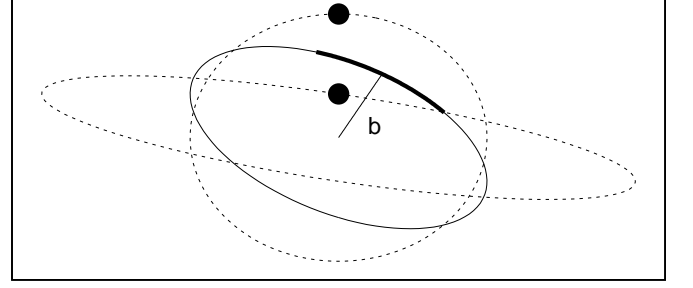


FIG. 1: Sketch of the orbits (dotted curves) of two “colliding” particles (which are plotted at the same arbitrary time). The plain curve represents their relative trajectory, and the thick portion (of length $\sim 2b$) the part of the trajectory in which $|\Delta \mathbf{V}_\perp|$ changes significantly (see text).

in the QSS (hypothesis which we have checked with our numerical simulations), the trajectories of the particles in the central region of the system (where collisional relaxation occurs) are ellipses. The relative motion of two particles is also therefore an ellipse which can be written as $\mathbf{r}(t) = (x_0 \cos(\omega t), y_0 \sin(\omega t))$, see Fig. 1. Because collisional relaxation is dominated by weak collisions, i.e., by the ones in which the trajectory of the particles are weakly perturbed, the change in relative velocity in the \hat{y} direction of two crossing particles can be well approximated by

$$|\Delta \mathbf{V}_\perp| \simeq g \int_0^{\pi/\omega} \frac{y_0 dt}{x_0^2 \sin^2(\omega t) + y_0^2 \cos^2(\omega t)} = \frac{g\pi}{\omega x_0}.$$

From geometrical arguments, it is possible to see that most of the orbits will have large ellipticity (for example, for elliptical orbits with (x_0, y_0) randomly distributed in a finite interval and random phases — which gives rise to an approximately homogeneous spatial distribution — we find numerically $y_0/x_0 \approx 0.3$ on average). If we choose the axis in order of $x_0 > y_0$, the distance of closest approach is y_0 and occurs at $t = 0$. As the integral converges rapidly an excellent approximation of (4) consists in taking as upper cutoff of the integral $\omega t \simeq y_0/x_0$. If $y_0/x_0 \ll 1$, and taking into account that collisions are very likely to happen in a central region of the QSS, we can therefore approximate the relative velocity during the collision by $|\mathbf{V}(t = 0)| \equiv V \simeq \omega x_0$. Then

$$|\Delta \mathbf{V}_\perp| \simeq \frac{g\pi}{V}, \quad (4)$$

where V is the relative velocity at the distance of closest approach. Interestingly, this is the same results than the one obtained in the spatial homogeneous case originally treated by Chandrasekhar applied to gravity in $d = 2$. In this study, it was considered a rectilinear relative trajectory with constant relative velocity V (e.g. [17]), in which

the distance of closest approach x_0 is the impact factor b . Then $|\Delta \mathbf{V}_\perp| \simeq 2 \int_0^\infty g/[b^2 + (Vt)^2] dt$ — which corresponds in this context to the change of velocity in the perpendicular direction to the relative velocity — gives exactly (4) (see Fig. 1). This result can be understood because an excellent approximation to the latter integral is obtained taking $t \simeq V/b$ as upper cutoff. Therefore we can conclude that the result (4) would be a good approximation when the curvature of the relative trajectories at the position of the distance of closest approach is small compared with the inverse of this distance. The arguments presented above apply also in $d = 3$, which may explain why the original Chandrasekhar estimate gives a good estimate of the relaxation time in inhomogeneous systems, taking as maximal impact parameter the size of the system (see e.g. [6, 13]).

We see that in the very particular case of two-dimensional gravity, the dependence on the impact parameter b does not appear in Eq. (4). This implies that, in the limit of validity of our approximations, all the particles contribute equally to the relaxation, independently of their distance. It is possible to compute the change in the relative parallel velocity using $|\Delta \mathbf{V}_\perp| = V \sin \theta$ and $|\Delta \mathbf{V}_\parallel| = V(1 - \cos \theta)$, where θ is the angle of deflection. In the weak collision approximation $\theta \ll 1$ and thus we have $\sin \theta \simeq \theta$ and $\cos \theta \simeq 1 - \theta^2/2$ and then $|\Delta \mathbf{V}_\parallel| = |\Delta \mathbf{V}_\perp|^2/2V$. Taking into account that particle masses are equal, we obtain for the change in velocity of a particle $|\Delta \mathbf{v}_\perp| \simeq \frac{\pi g}{2V}$ and $|\Delta \mathbf{v}_\parallel| \simeq \frac{\pi^2 g^2}{4V^2}$.

We compute the diffusion coefficients using the standard method used in $d = 3$; we will not detail the calculation here. To estimate in our case the number of collisions per unit of time, we see from our simulations that the velocity of the particles are approximately uncorrelated with their position. This allow us not to distinguish them. Moreover, we see also that the spatial density distribution is approximately constant up to a scale r^* (in radial coordinates). We can therefore estimate the number of collisions of a particle, on average, as $NV\Delta t/2r^*$. The mean field ($N \rightarrow \infty$) contribution of the change in velocity cancels out in average. Averaging over the velocity distribution, we obtain, keeping only terms of $\mathcal{O}(g^2)$ (see [17] app. L):

$$\begin{aligned} D_{v_i}(v) &= \frac{\langle \Delta v_i \rangle}{\Delta t} = C \frac{\partial q(v)}{\partial v_i} \\ D_{v_i v_j}(v) &= \frac{\langle \Delta v_i \Delta v_j \rangle}{\Delta t} = C \frac{\partial^2 p(v)}{\partial v_i \partial v_j}, \end{aligned} \quad (5)$$

where $C = \pi^2 g^2 N / 8r^*$ and we have introduced, as in the $d = 3$ case, the Rosenbluth potential [21] $q(v) = \int d^2 v' s(v')/|\mathbf{v} - \mathbf{v}'|$ and $p(v) = \int d^2 v' s(v')|\mathbf{v} - \mathbf{v}'|$ and we have assumed that the velocity distribution is isotropic. As the succession of QSS have an approximate polar symmetry, it is then useful to write Eq. (3) in polar coordinates. Considering that the Rosenbluth potentials are

isotropic, we get

$$\frac{\partial \tilde{s}}{\partial t} = \hat{C} \left\{ -\frac{\partial}{\partial \hat{v}} \left[\left(q'(\hat{v}) + \frac{p'(\hat{v})}{\hat{v}^2} \right) \tilde{s} \right] + \frac{1}{2} \frac{\partial^2}{\partial \hat{v}^2} [p''(\hat{v}) \tilde{s}] \right\}. \quad (6)$$

We have defined $\tilde{s}(\hat{v})$ as the velocity distribution in polar coordinates, the primes denotes derivation with respect to v and $\hat{v} = |\mathbf{v}|/v_*$. We define the velocity units v_* using the virial theorem, which states that, for any stationary state (and hence a QSS), the average velocity square of the particles is constant during all the evolution and it is $\langle v^2 \rangle = gN/2$ (e.g. [15]). It is then natural to take as velocity unit $v_* = \sqrt{gN}$. Equation (6) depends on N through the factor $\hat{C} = C/v_*^3$. Therefore $\hat{C} \sim (N\tau_{dyn})^{-1}$, which implies $\tau_{coll} \sim N\tau_{dyn}$, where we have defined the dynamical time of the system as $\tau_{dyn} = 1/\sqrt{gN}$.

To compute explicitly the diffusion coefficients we need an explicit form of $\tilde{s}(\hat{v})$. Taking as velocity distribution the equilibrium Gaussian one $\tilde{s}(\hat{v}) = 2\hat{v}v_*\beta \exp(-\beta\hat{v}^2)$ (with $\beta = 2$), because of virial theorem, we will make an error from only the fourth moment of the velocity. We obtain in this approximation

$$q(\hat{v}) = v_*^{-1} e^{-\beta\hat{v}^2} \sqrt{2\pi\beta} I_0(\beta\hat{v}^2) \quad (7a)$$

$$\begin{aligned} p(\hat{v}) &= v_* \sqrt{\frac{\pi}{8\beta}} e^{-\beta\hat{v}^2} \left[-e^{-\beta\hat{v}^2} + (1 + 2\beta\hat{v}^2) I_0(\beta\hat{v}^2) \right. \\ &\quad \left. + 2\beta\hat{v}^2 I_1(\beta\hat{v}^2) \right], \end{aligned} \quad (7b)$$

where $I_n(x)$ is the modified Bessel function of the first kind. It is possible to verify that $\tilde{s}(\hat{v}) = 2\hat{v}v_*\beta \exp(-\beta\hat{v}^2)$ is a stationary solution of Eq. (6) with the diffusion coefficients given by Eq. (7).

Numerical simulations.— We compare the theoretical model with molecular dynamics simulations performed with a modification of the publicly available code GADGET2 [22] to handle the logarithmic interaction. We use a time-step of $2.5 \times 10^{-4} \tau_{dyn}$ in order to ensure a very precise energy conservation, which is better than 10^{-5} for the whole duration of the runs. We performed simulations with initial water-bag conditions and initial virial ratio $\mu = v_*/\sqrt{2\langle v_0^2 \rangle} = 1$ and $\mu = 1.7$ (where $\langle v_0^2 \rangle$ is the average of the initial velocity square), and different number of particles in the interval $N = [100, 8000]$. The simulations have been performed for times between $t = 500\tau_{dyn}$ for the largest N and $t = 750\tau_{dyn}$ for the smallest ones. In order to improve statistics, we average the measured velocity distribution over 100 consecutive snapshots in an interval of $2.5\tau_{dyn}$. We monitor how the system approaches thermal equilibrium using the parameter $\xi(t) = \frac{1}{N^2} \int_0^\infty [N(v, t) - N_{MB}(v)]^2 dv$, where $N_{MB}(v)$ is the thermal equilibrium velocity distribution. In order to compare simulations with theory we compute the associated Langevin equation of Eq. (6). Therefore, the change in the velocity is given, following the Ito defini-

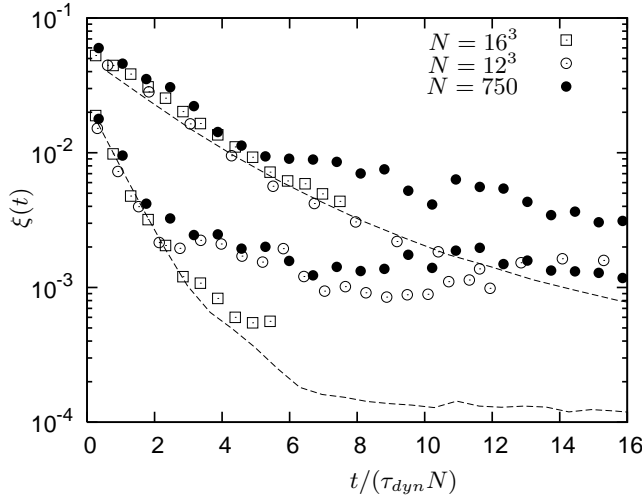


FIG. 2: Upper curves: initial condition with $\mu = 1$. Lower curves: initial conditions with $\mu = 1.7$. Points: evolution of the crossover parameter $\xi(t)$ measured in the molecular dynamics simulations for the two different initial condition $\mu = 1$ and $\mu = 1.7$. Lines: theoretical prediction calculated using Eq. (8) for each case (see text).

tion, by

$$dv(t) = \hat{C} \left\{ \left(q'(\hat{v}) + \frac{p'(\hat{v})}{\hat{v}^2} \right) dt + \sqrt{2p''(\hat{v})} dW \right\}, \quad (8)$$

where dW is a Gaussian stochastic variable delta correlated in time with variance unity. We choose as initial condition a configuration of the numerical simulation at $t/\tau_{dyn} = 260$ (time in which the system has violently relaxed) and then we compare the evolution predicted by the Langevin equation and the one of the full numerical simulation. We integrate Eq. (8) by a simple Euler procedure. In Fig. 2 we show the evolution of $\xi(t)$, where the time axis has been rescaled by a factor N , which indicates a scaling of the relaxation time as $\tau_{coll} \sim N\tau_{dyn}$. For clarity, between all the simulations with different numbers of particles performed we plot three of them. The part of the curve which flattens corresponds to thermal equilibrium, which is attained first as N decreases. The matching between the curves corresponding to different N is very good in the region out of equilibrium, as it has been illustrated for $N = 750$, $N = 12^3$ and $N = 16^3$, which confirm the prediction of Eq. (6) for the scaling of the relaxation. The dashed curves corresponds to the theoretical prediction given by Eq. (8) with $r^* = 0.38$ for the simulation with $\mu = 1$ and $r^* = 0.2$ for the simulation with $\mu = 1.7$. These values are close to the scale of the falloff in the density distribution; the density decays to half its center value around $r \approx 0.4$ for both set of simulations. Note that the difference in the slopes of the curves is essentially due to the different initial conditions considered for each case rather than in the value of r^*

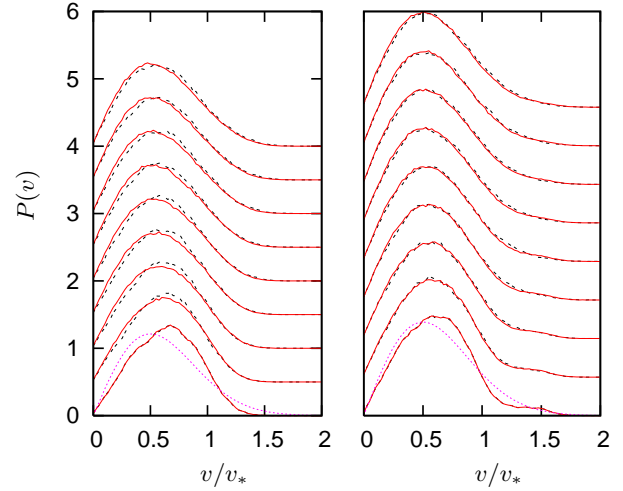


FIG. 3: Evolution of the velocity distribution function (left panel initial conditions $\mu = 1$ and right panel initial conditions $\mu = 1.7$) for the molecular dynamics simulation (straight red line) and the prediction of the Langevin equation (dotted blue line). The time increases as the shift in the y axis increases. At the last time the system has reached thermal equilibrium. The pink dashed line (plotted for the first time) corresponds to the distribution of thermal equilibrium.

taken. The full simulation curves decay to a lower value at thermal equilibrium because fluctuations appears to be larger in the molecular dynamics simulations than in the Langevin simulation. In Fig. 3 we show the evolution of the full velocity distribution function for both the simulation and the theory. We see a very good agreement between them, and specially in the $\mu = 1.7$ simulation, in which the Langevin equation is able to reproduce properly the relaxation of the core-halo structure in velocity space.

Conclusions. – We do not observe the scaling $\tau_{coll} \sim N^{1.35}\tau_{dyn}$ observed in [15]. This is due to the fact that their simplified dynamics appear not to describe properly the collisional dynamics of the real $d = 2$ system.

Some final remarks can be made about the maximal impact parameter which has to be considered in the calculations. In our calculations we don't use any artificial spatial cutoff in counting the number of collisions per unit of time. This is equivalent to not introducing any cutoff in the maximal impact parameter for interactions in which the change in velocity depends on it— such as the gravity in $d = 3$. We obtained a good agreement between theory and simulations not only in the scaling of τ_{coll} with N but also in the amplitude of $\xi(t)$, which allows us to conclude that it is not necessary to introduce any artificial cutoff in the calculations but integrals are automatically regularized by the size of the system. This result is in agreement with simulations performed in $d = 3$ dimensions [16] with potential interactions $u(r) \sim 1/r^\gamma$,

$\gamma \leq 1$, in which the maximal impact parameter to take in the Chandrasekhar approximation was numerically estimated to be $1/3$ the size of the system.

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